Lewis-Zagier Correspondence for higher order forms

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Abstract: The Lewis-Zagier correspondence, which attaches period functions to Maaß wave forms, is extended to wave forms of higher order, which are higher invariants of the Fuchsian group in question. The key ingredient is an identification of Higher order invariants with ordinary invariants of unipotent twists. This makes it possible to apply standard methods of automorphic forms to higher order forms.

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Introduction

The Lewis-Zagier correspondence [14, 15, 16], see also [2], is a bijection between the space of Maaß wave forms of a fixed Laplace-eigenvalue λ and

the space of real-analytic functions on the line satisfying a functional equation which involves the eigenvalue. The latter functions are called period functions. In [4] this correspondence has been extended to subgroups Γ of finite index in the full modular group $\Gamma(1)$. One can assume Γ to be normal in $\Gamma(1)$. The central idea of the latter paper is to consider the action of the finite group $\Gamma(1)/\Gamma$, and in this way to consider Maaß forms for Γ as vector-valued Maaß forms for $\Gamma(1)$. This technique can be applied to higher order forms [3, 5, 6, 7, 8, 9, 10] as well, turning the somewhat unfamiliar notion of a higher order invariant into the notion of a classical invariant of a twist by unipotent representation. This general framework is described in the first section of the present paper. This way of viewing higher order forms has the advantage that it allows techniques of classical automorphic forms to be applied in the context of higher order forms. The example of the trace formula will be subject of further investigations by the current author in the near future. In the present, we apply this technique to extend the Lewis-Zagier correspondence to higher order forms.

We define the corresponding spaces of automorphic forms of higher order in the second section. Holomorphic forms of higher order have been defined by various authors. Maaß forms are more subtle, as it is not immediately clear, how to establish the L^2-structure on higher order invariants. In the paper [6], the authors resorted to the obvious L^2-structure for the quotient spaces of consecutive higher order forms, which however is unsatisfactory because one whishes to view L^2-higher order forms as higher order invariants themselves. In the present paper this flaw is remedied, as we give a space of locally square-integrable functions on the universal cover of the Borel-Serre compactification whose higher order invariants give the sought for L^2-invariants. We also give a guide how to set up higher order L^2-invariants in more general cases, like lattices in locally compact groups, when there is no such gadget as the Borel-Serre compactification around. In the remaining sections we set up and verify the correspondence where we proceed roughly along the lines of the paper [4]. Note in paticular, that the correspondence is formulated in a uniform way for any order, so the distinction between ordinary forms and higher order forms comes only by the nature of the twist.

1 Higher order invariants and unipotent representations

In this section we describe higher order forms by means of invariants in unipotent representations.

Let Γ be a group and let W be a $\mathbb{C}[\Gamma]$-module. We here take the field $\mathbb{C}$ of complex numbers as base ring. Most of the general theory works over any ring, but our applications are over $\mathbb{C}$. Let $\boxed{I_\Gamma}$ be the augmentation ideal in $\mathbb{C}[\Gamma]$ this is the kernel of the *augmentation homomorphism* $A : \mathbb{C}[\Gamma] \to \mathbb{C}$; $\sum_\gamma c_\gamma \gamma \mapsto \sum_\gamma c_\gamma$. The ideal I_Γ is a vector space with basis $(\gamma - 1)_{\gamma \in \Gamma \smallsetminus \{1\}}$.

In the sequel, we will need two simple properties of the augmentation ideal which, for the convenience of the reader, we will prove in the following lemma. A set S of generators of the group Γ is called *symmetric*, if $s \in S \Rightarrow s^{-1} \in S$.

Lemma 1.1 (a) $\mathbb{C}[\Gamma] = \mathbb{C} \oplus I_\Gamma$.

(b) *For any given set of generators S of Γ one has*

$$I_\Gamma \;=\; \sum_{s \in S} \mathbb{C}[\Gamma](s-1).$$

Proof: (a) is clear, as is the inclusion "$\supset$" in (b). In order to show "$\subset$" we denote the right hand side as J and we have to show that $\gamma - 1 \in J$ for every $\gamma \in \Gamma$. As $\mathbb{C}[\Gamma](s-1) = \mathbb{C}[\Gamma]s^{-1}(s-1) = \mathbb{C}[\Gamma](s^{-1}-1)$, we may assume that S is symmetric. We proceed by induction on the word length of γ. We start by $\gamma \in S$ in which case the claim is clear. Next assume we have $\sigma \in \Gamma$ with $\sigma - 1 \in J$ and $s \in S$. We want to show that $\sigma s - 1 \in J$. For this note that

$$\sigma s - 1 \;=\; (\sigma - 1)(s-1) + (\sigma - 1) + (s-1).$$

As the right hand side of this equation is in J, so is the left. $\square$

We also fix a normal subgroup P of Γ. We let I_P denote the augmentation ideal of P and $\tilde{I}_P = \mathbb{C}[\Gamma]I_P$. As P is normal, $\tilde{I}_P$ is a two-sided ideal of $\mathbb{C}[\Gamma]$. For any integer $q \geq 0$ we set

$$J_q \;=\; I_\Gamma^q + \tilde{I}_P.$$

The set of Γ-invariants $W^\Gamma = H^0(\Gamma, W)$ in W can be described as the set of all $w \in W$ with $I_\Gamma w = 0$. For $q = 0, 1, 2, \dots$ we define the set of invariants of type P and order q to be

$$H^0_{q,P}(\Gamma, W) \; = \; \{w \in W : J_{q+1} w = 0\}.$$

Then $H^0_{q,P} = H^0_{q,P}(\Gamma, W)$ is a submodule of W and we have a natural filtration

$$0 \subset H^0_{0,P} \subset H^0_{1,P} \subset \cdots \subset H^0_{q,P} \subset \dots,$$

and as $I_\Gamma H^0_{q,P} \subset H^0_{q-1,P}$, the group Γ acts trivially on $H^0_{q,P}/H^0_{q-1,P}$.

A representation (η, V_η) of Γ on a complex vector space V_η is called a *unipotent length q representation*, if V_η has a Γ-stable filtration

$$0 \subset V_{\eta,0} \subset \cdots \subset V_{\eta,q} = V_\eta,$$

such that Γ acts trivially on each quotient $V_{\eta,k}/V_{\eta,k-1}$ where $k = 0, \dots, q$ and $V_{\eta,-1} = 0$.

Let a unipotent length q representation (η, V_η) be given. We also assume that it is P-trivial, i.e., the restriction to the subgroup P is the trivial representation. There is a natural map

$$\Phi_\eta : \mathrm{Hom}_\Gamma(V_\eta, W) \otimes V_\eta \to W$$

given by $\alpha \otimes v \mapsto \alpha(v)$.

Lemma 1.2 *Let W be a $\mathbb{C}[\Gamma]$-module. The submodule $H^0_{q,P}(\Gamma, W)$ constitutes a P-trivial, unipotent length q representation of Γ. If the group Γ is finitely generated, then the space $H^0_{q,P}(\Gamma, W)$ is the sum of all images Φ_η when η runs over the set of all P-trivial, unipotent length q representations which are finite dimensional over $\mathbb{C}$.*

Proof: The first assertion is clear. Assume now that Γ is finitely generated. The space $H^0_{q,P}(\Gamma, W)$ needn't be finite dimensional. We use induction on q to show that for each $w \in H^0_{q,\mathrm{par}}(\Gamma, W)$ the complex vector space $\mathbb{C}[\Gamma]v$ is finite dimensional. For $q = 0$ we have $\mathbb{C}[\Gamma]w = \mathbb{C}w$ and the claim follows. Next let $w \in H^0_{q+1,P}(\Gamma, W)$ and let S be a finite set of generators of Γ. Then

$$\mathbb{C}[\Gamma]w \; = \; \mathbb{C}w + I_\Gamma w \; = \; \mathbb{C}w + \sum_{s \in S} \mathbb{C}[\Gamma](s-1)w.$$

The element $(s-1)w$ lies in $H^0_{q,P}(\Gamma, W)$, so by induction hypothesis, the claim follows. $\square$

Assume from now on that Γ is finitely generated. The philosophy pursued in the rest of the paper is this:

Once you know $\mathrm{Hom}_\Gamma(V_\eta, W)$ *for every* P*-trivial,*
finite dimensional unipotent length q *representation,*
you know the space $H^0_{q,P}(\Gamma, W)$*.*

So, instead of investigating $H^0_{q,P}(\Gamma, W)$ one should rather look at

$$\mathrm{Hom}_\Gamma(V_\eta, W) \cong (V_\eta^* \otimes W)^\Gamma,$$

which is often easier to handle. In fact, it is enough to restrict to a generic set of η. As an example of this philosophy consider the case $q=1$. For each group homomorphism $\chi : \Gamma/P \to (\mathbb{C}, +)$ one gets a P-trivial, unipotent length q representation $\boxed{\eta_\chi}$ on $\mathbb{C}^2$ given by $\eta_\chi(\gamma) = \begin{pmatrix} 1 & \chi(\gamma) \\ & 1 \end{pmatrix}$.

We introduce the following notation

$$\bar H^0_{q,P} \;=\; \bar H^0_{q,P}(\Gamma, W) \;=\; H^0_{q,P}(\Gamma, W)/H^0_{q-1,P}(\Gamma, W) \;=\; H^0_{q,P}/H^0_{q-1,P}.$$

Proposition 1.3 *The space* $H^0_{1,P}(\Gamma, W)$ *is the sum over all images* Φ_{η_χ}*, where* χ *ranges in* $\mathrm{Hom}(\Gamma/P, \mathbb{C}) \smallsetminus \{0\}$*. For any two* $\chi \neq \chi'$ *one has*

$$\mathrm{Im}(\Phi_{\eta_\chi}) \cap \mathrm{Im}(\Phi_{\eta_{\chi'}}) \;=\; H^0(\Gamma, W).$$

In other words, one has

$$\bar H^0_1 \;=\; \bigoplus_\chi \mathrm{Im}(\Phi_{\eta_\chi})/H^0.$$

Proof: We make use of the *order lowering operator*

$$\Lambda : \bar H^0_{q,P} \to \mathrm{Hom}(\Gamma/P, \bar H^0_{q-1,P}) \cong \mathrm{Hom}(\Gamma/P, C) \otimes \bar H^0_{q-1,P},$$

where the last isomorphism is due to the fact that Γ is finitely generated. This operator is defined as

$$\Lambda(w)(\gamma) \;=\; (\gamma - 1)w.$$

One sees that this indeed is a homomorphism in γ by using the fact that

$$(\gamma\tau - 1) \equiv (\gamma - 1) + (\tau - 1) \mod I^2$$

for any two $\gamma, \tau \in \Gamma$. The map Λ is clearly injective.

Let now $w \in \mathrm{Im}(\Phi_{\eta_\chi}) \cap \mathrm{Im}(\Phi_{\eta_{\chi'}})$ for $\chi \neq \chi'$. Then

$$\Lambda(w) \in \chi \otimes H^0 \cap \chi' \otimes H^0,$$

and the latter space is zero as $\chi \neq \chi'$. For surjectivity, let $w \in H_1^0$. Then $\Lambda(w) = \sum_{i=1}^n \chi_i \otimes w_i$ with $w_i \in H^0$, and so $w \in \sum_{i=1}^n \mathrm{Im}(\phi_{\eta_{\chi_i}})$. $\square$

2 Higher order forms

We next define some spaces of automorphic forms of higher order, like holomorphic modular forms or Maaß wave forms [3, 5, 6, 7, 8, 9, 10]. The holomorphic case has been treated in various other places. Maaß forms are more subtle, as it is not immediately clear, how to establish the L^2-structure on higher order invariants. In the paper [6] the authors resorted to the obvious L^2-structure for the quotient spaces $\bar{H}^0_{q,P}$, which however is unsatisfactory because one whishes to view L^2-higher order forms as higher order invariants themselves. In the present paper this flaw is remedied, as we give a space of locally square-integrable functions on the universal cover of the Borel-Serre compactification whose higher order invariants give the sought for L^2-invariants. We also give a guide how to set up higher order L^2-invariants in more general cases, like general lattices in locally compact groups, when there is no such gadget as the Borel-Serre compactification around. In that case, Lemma 2.1 tells you how to define the L^2-structure once you have chosen a fundamental domain for the group action.

Let $\boxed{G}$ denote the group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$. It has the group $\boxed{K} = \mathrm{PSO}(2) = \mathrm{SO}(2)/\{\pm 1\}$ as a maximal compact subgroup. Let $\boxed{\Gamma(1)} = \mathrm{PSL}_2(\mathbb{Z})$ be the full modular group. Let $\boxed{\Gamma} \subset \Gamma(1)$ be a normal subgroup of finite index which is torsion-free. For every cusp c of Γ fix $\boxed{\sigma_c} \in \Gamma$ such that $\sigma_c\infty = c$ and $\sigma_c^{-1}\Gamma_c\sigma_c = \pm\begin{pmatrix} 1 & N_c\mathbb{Z} \\ & 1 \end{pmatrix}$. The number $\boxed{N_c} \in \mathbb{N}$ is uniquely determined and is called the *width* of the cusp c. Let $\boxed{\mathbb{H}} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half plane and let $\boxed{\mathcal{O}(\mathbb{H})}$ be the set of holomorphic functions

on $\mathbb{H}$. We fix a weight $k \in 2\mathbb{Z}$ and define a (right-) action of G on functions f on $\mathbb{H}$ by

$$f|_k\gamma(z) \;=\; (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We define the space $\boxed{\mathcal{O}^M_{\Gamma,k}(\mathbb{H})}$ to be the set of all $f \in \mathcal{O}(\mathbb{H})$, such that for every cusp c of Γ the function $f|_k\sigma_c$ is, in the domain $\{\mathrm{Im}(z) > 1\}$, bounded by a constant times $\mathrm{Im}(z)^A$ for some $A > 0$.

Further we consider the space $\boxed{\mathcal{O}^S_{\Gamma,k}(\mathbb{H})}$ of all $f \in \mathcal{O}(\mathbb{H})$, such that for every cusp c of Γ the function $f|_k\sigma_c$ is, in the domain $\{\mathrm{Im}(z) > 1\}$, bounded by a constant times $e^{-A\mathrm{Im}(z)}$ for some $A > 0$.

These two spaces are preserved not only by Γ, but also by the action of the full modular group $\Gamma(1)$.

The normal subgroup P of Γ will be the subgroup $\boxed{\Gamma_{\mathrm{par}}}$ generated by all parabolic elements. We then write $H^0_{q,\mathrm{par}}$ for $H^0_{q,P}$. We consider the space of modular functions of weight k and order q,

$$\boxed{M_{k,q}(\Gamma)} \;=\; H^0_{q,\mathrm{par}}(\Gamma, \mathcal{O}^M_{\Gamma,k}(\mathbb{H})),$$

as well as the corresponding space of cusp forms

$$\boxed{S_{k,q}(\Gamma)} \;=\; H^0_{q,\mathrm{par}}(\Gamma, \mathcal{O}^S_{\Gamma,k}(\mathbb{H})).$$

Then every $f \in M_{k,q}(\Gamma)$ possesses a *Fourier expansion* at every cusp c of the form

$$f|_k\sigma_c(z) \;=\; \sum_{n=0}^{\infty} a_{c,n}\, e^{2\pi i \frac{n}{N_c} z}.$$

A function $f \in M_{k,q}(\Gamma)$ belongs to the subset $S_{k,q}(\Gamma)$ if and only if $a_{c,0} = 0$ for every cusp c of Γ.

As the group Γ is normal in $\Gamma(1)$, the latter group acts on the finite dimensional spaces $M_{k,q}(\Gamma)$ and $S_{k,q}(\Gamma)$. These therefore give examples of finite dimensional representations of $\Gamma(1)$ which become unipotent length q when restricted to Γ.

Recall that a *Maaß wave form* for the group Γ and parameter $\nu \in \mathbb{C}$ is a function $u \in L^2(\Gamma\backslash\mathbb{H})$ which is twice continuously differentiable and satisfies

$$\Delta u \;=\; \left(\tfrac{1}{4} - \nu^2\right) u.$$

By the regularity of solutions of elliptic differential equations this condition implies that u is real analytic. Let $\boxed{\mathcal{M}_\nu} = \mathcal{M}_\nu(\Gamma)$ be the space of all Maaß wave forms for Γ.

Next we define Maaß-wave forms of higher order. First we need the higher order version of the Hilbert space $L^2(\Gamma\backslash\mathbb{H})$. For this recall the construction of the Borel-Serre compactification $\overline{\Gamma\backslash\mathbb{H}}$ of $\Gamma\backslash\mathbb{H}$, see [1]. First one constructs a space $\boxed{\mathbb{H}_\Gamma} \supset \mathbb{H}$ by attaching to each cusp c of Γ a real line to $\mathbb{H}$ and then one equips this set with a suitable topology such that Γ acts properly discontinuously and the quotient $\Gamma\backslash\mathbb{H}_\Gamma$ is the Borel-Serre compactification. The space $\mathbb{H}_\Gamma$ is constructed in such a way, that for a given (closed) fundamental domain $D \subset \mathbb{H}$ of $\gamma\backslash\mathbb{H}$ which has finitely many geodesic sides, the closure $\overline{D}$ in $\mathbb{H}_\Gamma$ is a fundamental domain for $\Gamma\backslash\mathbb{H}_\Gamma$. By the discontinuity of the group action, this has the following consequence: For every compact det $K \subset \mathbb{H}_\Gamma$ there exists a finite set $F \subset \Gamma$ such that $K \subset F\overline{D} = \bigcup_{\gamma\in F}\gamma\overline{D}$.

Now we extend the hyperbolic measure to $\mathbb{H}_\Gamma$ in such a way that the boundary $\partial\mathbb{H}_\Gamma = \mathbb{H}_\Gamma \smallsetminus \mathbb{H}$ is a nullset. Let $\boxed{L^2_{\mathrm{loc}}(\mathbb{H}_\Gamma)}$ be the space of local L^2-functions on $\mathbb{H}_\Gamma$. Then Γ acts on $L^2_{\mathrm{loc}}(\mathbb{H}_\Gamma)$. Since Γ acts discontinuously with compact quotient on $\mathbb{H}_\Gamma$, one has

$$L^2_{\mathrm{loc}}(\mathbb{H}_\Gamma)^\Gamma \;=\; L^2(\Gamma\backslash\mathbb{H}).$$

Define the space $\boxed{L^2_q(\Gamma\backslash\mathbb{H})}$ as the space of all $f \in L^2_{\mathrm{loc}}(\mathbb{H}_\Gamma)$ such that $J_{q+1}f = 0$, so in other words,

$$L^2_q(\Gamma\backslash\mathbb{H}) \;=\; H^0_{q,\mathrm{par}}\left(\Gamma, L^2_{\mathrm{loc}}(\mathbb{H}_\Gamma)\right).$$

Then $L^2_0(\Gamma\backslash\mathbb{H}) = L^2(\Gamma\backslash\mathbb{H})$ is a Hilbert space in a natural way. We want to instal Hilbert space structures on the spaces $L^2_q(\Gamma\backslash\mathbb{H})$ for $q \geq 1$ as well. For this purpose we introduce the space $\boxed{F_q}$ of all measurable functions $f : \mathbb{H} \to \mathbb{C}$ such that $J_{q+1}f = 0$ modulo nullfunctions. Then $L^2_q(\Gamma\backslash\mathbb{H})$ is a subset of F_q.

Lemma 2.1 *Let $S \subset \Gamma$ be a finite set of generators which is assumed to be symmetric and to contain the unit element. Let $D \subset \mathbb{H}$ be a closed fundamental domain of Γ which has finitely many geodesic sides.*

Any $f \in F_q$ is uniquely determined by its restriction to

$$S^qD \;=\; \bigcup_{s_1,\dots,s_q\in S} s_1\dots s_qD.$$

One has

$$F_q \cap L^2(S^q D) \;=\; L^2_q(\Gamma\backslash\mathbb{H}),$$

where on both sides we mean the restriction to $S^q D$, which is unambiguous by the first assertion. In this way the space $L^2_q(\Gamma\backslash\mathbb{H})$ is a closed subspace of the Hilbert space $L^2(S^q D)$. The induced Hilbert space topology on $L^2_q(\Gamma\backslash\mathbb{H})$ is independent of the choices of S and D, although the inner product is not. The action of the group $\Gamma(1)$ on $L^2_q(\Gamma\backslash\mathbb{H})$ is continuous, but not unitary unless $q = 0$.

Proof: We have to show that any $f \in L_q = L^2_q(\Gamma\backslash\mathbb{H})$ which vanishes on $S^q D$, is zero. We use induction on q. The case $q = 0$ is clear. Let $q \geq 1$ and write $\boxed{\bar{L}_q} = L_q/L_{q-1}$. Consider the *order lowering operator*

$$\Lambda : L_q \to \mathrm{Hom}(\Gamma, \bar{L}_{q-1}) \cong \mathrm{Hom}(\Gamma, \mathbb{C}) \otimes \bar{L}_{q-1},$$

given by

$$\Lambda(f)(\gamma) \;=\; (\gamma - 1)f.$$

The kernel of Λ is L_{q-1}. Now assume $f(S^q D) = 0$. Then for every $s \in S$ we have $(s-1)f(S^{q-1}D) = 0$ and hence, by induction hypothesis, we conclude $(s-1)f = 0$. But as S generates Γ this means that $\Lambda(f) = 0$ and so $f \in L_{q-1}$, so, again by induction hypothesis, we get $f = 0$.

We next show that

$$F_q \cap L^2(S^q D) \;=\; F_q \cap L^2(S^{q+j} D)$$

for every $j \geq 0$. The inclusion "$\supset$" is clear. We show the other inclusion by induction on q and j. For $q = 0$ or $j = 0$ there is no problem. So assume the claim proven for $q - 1$. Let $f \in F_q \cap L^2(S^{q+j}D)$ and let $s \in S$. Then $f(sz) = f(z) + f(sz) - f(z)$, the function $f(z)$ in in $L^2(S^{q+j}D)$ and the function $f(sz) - f(z)$ is in $F_{q-1} \cap L^2(S^{q+j}D) \subset L^2(S^{q+j+1}D)$ by induction hypothesis. It follows that $f \in L^2(sS^{q+j}D)$ and since this holds for every s we get $f \in L^2(S^{q+j+1}D)$ as claimed.

We now come to

$$F_q \cap L^2(S^q D) \;=\; L^2_q(\Gamma\backslash\mathbb{H}).$$

Let $f \in F_q \cap L^2(S^q D)$. For every compact subset K of $\mathbb{H}_\Gamma$ there exists $j \geq 0$ such that $K \subset S^{q+j}D$. Therefore, f is in $L^2(K)$ for every compact subset K of H_Γ. As the latter space is locally compact, f is in $L^2_{\mathrm{loc}}(\mathbb{H}_\Gamma)$. Since

$I_\Gamma^{q+1}f = 0$ we get $f \in L^2_q(\Gamma\backslash\mathbb{H})$. For the other inclusion let $f \in L^2_q(\Gamma\backslash\mathbb{H})$. As S^qD is relatively compact in $\mathbb{H}_\Gamma$ it follows that $f \in L^2(S^qD)$ as claimed.

We next show independence of the topology of S. So let S' be another set of generators. Then there exists $l \in \mathbb{N}$ such that $S' \subset S^l$. Hence ist suffices to show that the topology from the inclusion $L^2_q(\Gamma\backslash\mathbb{H}) \subset L^2(S^qD)$ coincides with the topology from the inclusion $L^2_q(\Gamma\backslash\mathbb{H}) \subset L^2(S^{q+j}D)$ for every $j \geq 0$. If a sequence tends to zero in the latter, it clearly tends to zero in the first. The other way round is proven by induction on j similar to the above. In particular, the continuity of the $\Gamma(1)$-action follows.

Finally, we show the independence of D. Let D' be another closed fundamental domain with finitely many geodesic sides. Then there exists $l \in \mathbb{N}$ such that $D' \subset S^lD$ and the claim follows along the same lines as above.

□

We define the space $\boxed{\mathcal{M}_{\nu,q}} = \mathcal{M}_{\nu,q}(\Gamma)$ of *Maaß-wave forms of order* q to be the space of all $u \in L^2_q(\Gamma\backslash\mathbb{H})$ which are twice continuously differentiable and satisfy

$$\Delta u \;=\; \left(\frac{1}{4} - \nu^2\right) u.$$

Fix a finite dimensional representation (η, V_η) of $\Gamma(1)$,
which is Γ_{par}-trivial and becomes a unipotent
length q representation on restriction to Γ.

We set $\boxed{\mathcal{M}_{\nu,q,\eta}}$ equal to $(V_\eta \otimes \mathcal{M}_{\nu,q})^{\Gamma(1)}$. Likewise we define $\boxed{\tilde{\mathcal{M}}_{\nu,q}(\Gamma)}$ as the space of all $u \in F_q(\Gamma)$ which are twice continuously differentiable and satisfy $\Delta u = \left(\frac{1}{4} - \nu^2\right) u$, and we set $\boxed{\tilde{\mathcal{M}}_{\nu,q,\eta}} = (V_\eta \otimes \tilde{\mathcal{M}}_{\nu,q})^{\Gamma(1)}$.

Lemma 2.2 *Let* $\boxed{\mathcal{D}'_\nu}$ *be the space of all distributions* u *on* $\mathbb{H}$ *with* $\Delta u = (\frac{1}{4} - \nu^2)u$. *Then*

$$\tilde{\mathcal{M}}_{\nu,q,\eta} \;=\; (V_\eta \otimes \tilde{\mathcal{M}}_{\nu,q})^{\Gamma(1)} \;=\; (V_\eta \otimes \mathcal{D}'_\nu)^{\Gamma(1)}$$

and

$$\mathcal{M}_{\nu,q,\eta} \;=\; (V_\eta \otimes \mathcal{M}_{\nu,q})^{\Gamma(1)} \;=\; \left(V_\eta \otimes (\mathcal{D}'_\nu \cap L^2_{\mathrm{loc}}(\mathbb{H}_\Gamma))\right)^{\Gamma(1)}.$$

Proof: The inclusion "$\subset$" is obvious in both cases. We show "$\supset$". In the first case, the space on the left can be described as the space of all smooth functions $u : \mathbb{H} \to V_\eta$ satisfying $\Delta u = (\frac{1}{4} - \nu^2)u$ as well as $J_{q+1}u = 0$ and $u(\gamma z) = \eta(\gamma)u(z)$ for every $\gamma \in \Gamma(1)$. Now let $u \in (V_\eta \otimes \mathcal{D}'_\nu)^{\Gamma(1)}$. As u satisfies an elliptic differential equation with smooth coefficients, u is a smooth function with $\Delta u = (\frac{1}{4} - \nu^2)u$. The condition $u(\gamma z) = \eta(\gamma)u(z)$ is clear. Finally, the condition $J_{q+1}u = 0$ follows from that, as $\eta|_\Gamma$, being Γ_{par}-trivial and unipotent of length q, satisfies $\eta(J_{q+1}) = 0$. Hence the first claim is proven. The second is similar. $\square$

As in the holomorphic case, every Maaß-form $f \in \mathcal{M}_{\nu,q}(\Gamma)$ has a Fourier expansion at every cusp c,

$$f(\sigma_c z) \;=\; \sum_{n=0}^{\infty} a_{c,n}(y)\, e^{2\pi i \frac{n}{N_c} x},$$

with smooth functions $a_{c,n}(y)$.

We define the space $\boxed{\mathcal{S}_{\nu,q}}$ of *Maaß cusp forms* to be the space of all $f \in \mathcal{M}_{\nu,q}$ with $a_{c,0}(y) = 0$ for every cusp c. We also set $\boxed{\mathcal{S}_{\nu,q,\eta}} = (V_\eta \otimes \mathcal{S}_{\nu,q})^{\Gamma(1)}$.

Note that since η is unipotent of length q on Γ, we have

$$\mathcal{S}_{\nu,q,\eta} \;=\; \left(V_\eta \otimes \mathcal{S}_{\nu,q'}\right)^{\Gamma(1)}$$

for every $q' \geq q$.

3 Setting up the transform

It is the aim of this note to extend the Lewis Correspondence [4, 14, 15, 16] to the case of higher order forms. We will explain the approach in the case of cusp forms first.

Throughout, let (η, V_η) be a finite dimensional representation of $\Gamma(1)$ which becomes Γ_{par}-trivial and unipotent of length q when restricted to Γ.

We fix the following notation for the canonical generators of $\Gamma(1)$:

$$S \;=\; \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad T \;=\; \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then $S^2 = \mathbf{1} = (ST)^3$, and T is of infinite order. Note that there exists $N \in \mathbb{N}$ such that $T^N \in \Gamma$ as $\Gamma(1)/\Gamma$ is a finite group. Let N be minimal with this property, then $N = N_\infty$ is the width of the cusp ∞ of Γ. We then have $\eta(T)^N = \eta(T^N) = 1$, as η is trivial on parabolic elements of Γ.

Let $\boxed{\mathcal{F}_\eta}$ be the space of holomorphic functions $f\colon \mathbb{C} \smallsetminus \mathbb{R} \to V_\eta$ with

$$\begin{aligned} f(z+1) &= \eta(T)f(z), \\ f(z) &= O(1) \qquad \text{as } |\mathrm{Im}(z)| \to \infty, \\ f(i\infty) + f(-i\infty) &= 0. \end{aligned}$$

The last condition needs explaining. Let $N = N_\infty$ be the width of the cusp ∞, Then f has a Taylor expansion

$$f(z) \;=\; \sum_{k\in\mathbb{Z}} e^{2\pi i \frac{k}{N} z} v_k^+, \qquad v_k^+ \in V_\eta,$$

in $\mathbb{H}$ and similarly with $v_k^- \in V_\eta$ in the lower half plane $\boxed{\bar{\mathbb{H}}}$. The boundedness condition leads to $v_{-k}^+ = v_k^- = 0$ for every $k \in \mathbb{N}$. So we write $f(\pm i\infty)$ for the vector $v_0^\pm$ and the condition then just means that $v_0^+ + v_0^- = 0$.

For a complex number ν consider the space $\boxed{\mathcal{F}_{\nu,\eta}}$ of all $f \in \mathcal{F}_\eta$ for which the map

$$z \mapsto f(z) - z^{-2\nu-1}\,\eta(S) f\left(\frac{-1}{z}\right)$$

extends holomorphically to $\mathbb{C} \smallsetminus (-\infty, 0]$. Here $z^{-2\nu-1}$ is defined by

$$z^{-2\nu-1} \;=\; \exp\left((-2\nu-1)\log z\right),$$

where $\log z$ stands for the principal branch of the logarithm, i.e., the one that maps $\mathbb{C} \smallsetminus (-\infty, 0]$ to $\mathbb{R} + i(-\pi, \pi)$.

Let $\boxed{\Psi_{\nu,\eta}}$ be the space of all holomorphic functions $\psi\colon \mathbb{C} \smallsetminus (-\infty, 0] \to V_\eta$ satisfying the *Lewis equation*

$$\eta(T)\psi(z) = \psi(z+1) + (z+1)^{-2\nu-1}\eta(ST^{-1})\psi\left(\frac{z}{z+1}\right) \tag{1}$$

and the *asymptotic formula*

$$\begin{aligned} 0 \;=\; & e^{+\pi i\nu} \lim_{\mathrm{Im}(z)\to\infty} \left(\psi(z) + z^{-2\nu-1}\eta(S)\psi\left(\frac{-1}{z}\right)\right) + \\ & + e^{-\pi i\nu} \lim_{\mathrm{Im}(z)\to-\infty} \left(\psi(z) + z^{-2\nu-1}\eta(S)\psi\left(\frac{-1}{z}\right)\right), \end{aligned} \tag{2}$$

where both limits are supposed to exist.

Let $\boxed{A}$ denote the subgroup of G consisting of diagonal matrices and let $\boxed{N}$ be the subgroup of upper triangular matrices with ± 1 on the diagonal. The group G then as a manifold is a direct product $G = ANK$. For $\nu \in \mathbb{C}$ and $a = \pm\mathrm{diag}(t, t^{-1}) \in A$, $t > 0$, let $a^\nu = t^{2\nu}$. We insert the factor 2 for compatibility reasons.

Let $\boxed{(\pi_\nu, V_{\pi_\nu})}$ denote the principal series representation of G with parameter ν. The representation space V_{π_ν} is the Hilbert space of all functions $\varphi\colon G \to \mathbb{C}$ with $\varphi(anx) = a^{\nu+\frac{1}{2}}\varphi(x)$ for $a \in A, n \in N, x \in G$, and $\int_K |\varphi(k)|^2\, dk < \infty$ modulo nullfunctions. The representation is $\pi_\nu(x)\varphi(y) = \varphi(yx)$. There is a special vector $\boxed{\varphi_0}$ in V_{π_ν} given by

$$\varphi_0(ank) \;=\; a^{\nu+\frac{1}{2}}.$$

This vector is called the *basic spherical function* with parameter ν.

For a continuous G-representation (π, V_π) on a topological vector space V_π let $\boxed{\pi^\omega}$ denote the subrepresentation on the space of analytic vectors, i.e. V_{π^ω} consists of all vectors v in V_π such that for every continuous linear map $\alpha\colon V_\pi \to \mathbb{C}$ the map $g \mapsto \alpha(\pi(g)v)$ is real analytic on G. This space comes with a natural topology. Let $\boxed{\pi^{-\omega}}$ be its topological dual. In the case of $\pi = \pi_\nu$ it is known that π_ν^ω and $\pi_\nu^{-\omega}$ are in perfect duality, i.e., they are each other's topological duals. The vectors in $\pi_\nu^{-\omega}$ are called *hyperfunction vectors* of the representation π_ν.

As a crucial tool we will use the space

$$\boxed{A_{\nu,\eta}^{-\omega}} = (\pi_\nu^{-\omega} \otimes \eta)^{\Gamma(1)} = H^0(\Gamma(1), \pi_\nu^{-\omega} \otimes \eta)$$

and call it the space of *η-automorphic hyperfunctions.*

Generalizing results of Bruggeman (see [2], Prop. 2.1 and Prop. 2.3), we will show in Proposition 4.2 that there is a linear isomorphism $A_{\nu,\eta}^{-\omega} \to \mathcal{F}_{\nu,\eta}$ and (using this) establish in Proposition 4.4 a linear map

$$B\colon A_{\nu,\eta}^{-\omega} \to \Psi_{\nu,\eta},$$

which we call the *Bruggeman transform.* It turns out to be bijective unless $\nu \in \frac{1}{2} + \mathbb{Z}$.

For $\mathrm{Re}(\nu) > -\frac{1}{2}$ consider the space $\boxed{\Psi_{\nu,\eta}^o}$ of all $\psi \in \Psi_{\nu,\eta}$ satisfying

$$\psi(z) \;=\; O(\min\{1, |z|^{-C}\}) \qquad \text{for } z \in \mathbb{C} \setminus (-\infty, 0], \tag{3}$$

for some $0 < C < 2\mathrm{Re}(\nu)+1$. We call the elements of $\Psi^o_{\nu,\eta}$ *period functions.*

For an automorphic hyperfunction $\alpha \in A^{-\omega}_{\nu,\eta}$ we consider the function $u\colon G \to V_\eta$ given by

$$u(g) \quad \stackrel{\text{def}}{=} \quad \langle \pi_{-\nu}(g)\varphi_0, \alpha\rangle \,.$$

Here $\langle\cdot,\cdot\rangle$ is the canonical pairing $\pi^\omega_{-\nu} \times \pi^{-\omega}_{-\nu} \otimes \eta \to V_\eta$. Then u is right K-invariant, hence can be viewed as a function on $\mathbb{H}$. As such it lies in $\tilde{\mathcal{M}}_{\nu,\eta}$ since α is Γ-equivariant and the Casimir operator on G, which induces Δ, is scalar on π_ν with eigenvalue $\frac{1}{4} - \nu^2$. The transform $\boxed{P}\colon \alpha \mapsto u$ is called the *Poisson transform.* It follows from [17], Theorem 5.4.3, that the Poisson transform

$$P\colon A^{-\omega}_{\nu,\eta} \;\to\; \tilde{\mathcal{M}}_{\nu,\eta}$$

is an isomorphism for $\nu \notin \frac{1}{2} + \mathbb{Z}$.

For $\nu \notin \frac{1}{2}+\mathbb{Z}$ we finally define the *Lewis transform* as the map $\boxed{L}\colon \mathcal{M}_{\nu,\eta} \to \Psi_{\nu,\eta}$, given by

$$L \quad \stackrel{\text{def}}{=} \quad B \circ P^{-1}.$$

Our main result (see Theorem 5.3) is a generalization of [16], Thm. 1.1, and says that the Lewis transform for $\nu \notin \frac{1}{2} + \mathbb{Z}$ and $\mathrm{Re}(\nu) > -\frac{1}{2}$ restricts to a linear isomorphism between the space of Maaß cusp forms $\mathcal{S}_{\nu,\eta}$ and the space $\Psi^o_{\nu,\eta}$ of period functions.

A holomorphic function on $\mathbb{C} \smallsetminus (-\infty, 0]$ is uniquely determined by its values in $\mathbb{R}^+ = (0,\infty)$. Thus, in principle, it is possible to describe the period functions as a space of real analytic functions on the positive halfline. Following ideas from [16], Chap III, we show how this can be done in an explicit way.

We set $\boxed{\kappa_\nu} = \; -\min(0, 2\mathrm{Re}(\nu))$. Consider the space $\boxed{\Psi^{\mathbb{R}}_{\nu,\eta}}$ of all real analytic functions ψ from $(0,\infty)$ to V_η satisfying

$$\begin{aligned} \eta(T)\psi(x) &= \psi(x+1) + (x+1)^{-2\nu-1}\eta(ST^{-1})\psi\left(\frac{x}{x+1}\right) && (4)\\ \psi(x) &= o(x^{-1+\kappa_\nu}), \quad \text{as } x \to 0, x > 0, && (5)\\ \psi(x) &= o(x^{\kappa_\nu}), \quad \text{as } x \to +\infty, x \in \mathbb{R}. && (6) \end{aligned}$$

Our second main result (see Theorem 6.4) is a generalization of [16], Thm. 2, and says that for $\mathrm{Re}(\nu) > -\frac{1}{2}$ we have $\Psi^{\mathbb{R}}_{\nu,\eta} = \{\psi|_{(0,\infty)} : \psi \in \Psi^o_{\nu,\eta}\}$.

We summarize the various spaces and mappings considered so far in one diagram:

$$\begin{tikzcd}
\mathcal{M}_{\nu,\eta} \arrow[d] & \mathcal{S}_{\nu,\eta} \arrow[l] \arrow[r, "L"] \arrow[d] & \Psi^{o}_{\nu,\eta} \arrow[r, "\mathrm{res}"] \arrow[d] & \Psi^{\mathbb{R}}_{\nu,\eta} \\
\tilde{\mathcal{M}}_{\nu,\eta} & \mathcal{F}_{\nu,\eta} \arrow[r] & \Psi_{\nu,\eta} & \\
 & A^{-\omega}_{\nu,\eta} \arrow[ul, "P"] \arrow[u, "\cong"'] \arrow[ur, "B"'] & &
\end{tikzcd}$$

4 Automorphic hyperfunctions

The group G acts on the complex projective line $\mathbb{P}_1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ by linear fractional transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$, where $\mathbb{C}$ is embedded into $\mathbb{P}_1(\mathbb{C})$ via $z \mapsto [z : 1]$. This action has three orbits: the upper half plane $\mathbb{H}$, the lower half plane $\bar{\mathbb{H}}$ and the real projective line $\mathbb{P}_1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$. The upper half plane can be identified with G/K via $gK \mapsto g \cdot i$. Moreover, using

$$[r : s] \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\mathrm{def}}{=} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot \frac{r}{s} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \cdot \frac{r}{s} = [dr - bs : -cr + as],$$

the projective space $\mathbb{P}_1(\mathbb{R})$ can be identified with $AN\backslash G \cong [1 : 0] \cdot G$ via

$$AN \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [d : -c].$$

Under this identification $\mathbb{R} = AN\backslash ANwN$ for the Weyl group element $w = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ since

$$[1 : 0] \cdot w \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = [1 : 0] \cdot \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} = [x : 1].$$

Note that $ANwN$ is an open Bruhat cell in G. Therefore, if we set $n_x = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}$, then

$$\varphi(anwn_x) = f(x)$$

defines a realization of π_ν on functions on $\mathbb{R}$. Using the Bruhat decomposition

$$\pm \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \pm \begin{pmatrix} \gamma^{-1} & \alpha \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -\frac{\delta}{\gamma} \end{pmatrix}$$

for $\gamma \neq 0$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we find

$$\pi_\nu(g)\varphi \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} = (cx-a)^{-2\nu-1} \varphi \begin{pmatrix} 0 & 1 \\ -1 & \frac{dx-b}{-cx+a} \end{pmatrix}$$

and $\|\varphi\|^2 = \int_{\mathbb{R}} \left| \varphi \begin{pmatrix} 0 & 1 \\ -1 & -x \end{pmatrix} \right|^2 \frac{(1+x^2)^{2\nu}}{\pi} dx$. Thus we obtain a realization of V_{π_ν} on $L^2(\mathbb{R}, \frac{1}{\pi}(1+x^2)^{2\nu} dx)$ with the action

$$\pi_\nu \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = (cx-a)^{-2\nu-1} f\left(\frac{dx-b}{-cx+a} \right).$$

Transferring the action to $L^2(\mathbb{R}, \frac{1}{\pi}\frac{dx}{1+x^2})$ via

$$f(x) \leftrightarrow \tilde{f}(x) \stackrel{\text{def}}{=} (1+x^2)^{\nu+\frac{1}{2}} f(x)$$

then yields the action

$$\pi_\nu \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tilde{f}(x) = \left(\frac{1+x^2}{(cx-a)^2+(dx-b)^2} \right)^{\nu+\frac{1}{2}} \tilde{f}\left(\frac{dx-b}{-cx+a} \right)$$

used in [2]. This is the realization of the principal series we shall work with. Note that in this realization the basic spherical function is simply the constant function 1.

Let $\pi_\nu^\omega \subset \pi_\nu^{-\omega}$ be the sets of analytic vectors and hyperfunction vectors, respectively. For any open neighbourhood U of $\mathbb{P}_1(\mathbb{R})$ inside $\mathbb{P}_1(\mathbb{C})$ the space $\pi_\nu^{-\omega}$ can be identified with the space

$$\mathcal{O}(U \smallsetminus \mathbb{P}_1(\mathbb{R}))/\mathcal{O}(U),$$

where $\mathcal{O}$ denotes the sheaf of holomorphic functions. This space does not depend on the choice of U. For $U \subseteq \mathbb{C}$ this follows from Lemma 1.1.2 of [17] and generally by subtracting the Laurent series at infinity. The G-action is given by the above formula, where x is replaced by a complex variable z. Note that any hyperfunction α on $\mathbb{P}_1(\mathbb{R})$ has a restriction to $\mathbb{R}$ which can be represented by a holomorphic function on $\mathbb{C} \smallsetminus \mathbb{R}$.

Proposition 4.1 (Symmetry of gluing conditions) *For $f \in \mathcal{F}_\eta$ the following conditions are equivalent:*

(1) $z \mapsto f(z) - z^{-2\nu-1}\,\eta(S) f\left(\frac{-1}{z}\right)$ *extends holomorphically to* $\mathbb{C} \smallsetminus (-\infty, 0]$.

(2) $z \mapsto (1+z^{-2})^{\nu+\frac{1}{2}} f\left(\frac{-1}{z}\right)$ *and* $z \mapsto (1+z^2)^{\nu+\frac{1}{2}}\,\eta(S) f(z)$ *define the same hyperfunction on* $\mathbb{R} \smallsetminus \{0\}$.

Proof: "(2)⇒(1)" Suppose that

$$(1+z^{-2})^{\nu+\frac{1}{2}} f\left(\frac{-1}{z}\right) = (1+z^2)^{\nu+\frac{1}{2}}\,\eta(S) f(z) + q(z)$$

with a function q that is holomorphic in a neighborhood of $\mathbb{R} \smallsetminus \{0\}$. For $\mathrm{Re}(z) > 0$ we can divide the equation by $(1+z^2)^{\nu+\frac{1}{2}}$ and obtain

$$z^{-2\nu-1} f\left(\frac{-1}{z}\right) = \eta(S) f(z) + (1+z^2)^{-\nu-\frac{1}{2}}\, q(z).$$

Since $\eta(S) = \eta(S)^{-1}$, this implies the claim.

"(1)⇒(2)" If (1) holds, by the same calculation as above we see that for $\mathrm{Re}(z) > 0$ the function

$$z \mapsto (1+z^2)^{\nu+\frac{1}{2}}\,\eta(S) f(z) - (1+z^{-2})^{\nu+\frac{1}{2}} f\left(\frac{-1}{z}\right)$$

extends holomorphically to the entire right halfplane. But then the symmetry of this expression under the transformation $z \mapsto -\frac{1}{z}$ yields the holomorphic extendability also on the left halfplane which proves (2). □

Recall the space $A_{\nu,\eta}^{-\omega} = (\pi_\nu^{-\omega} \otimes \eta)^{\Gamma(1)} = H^0(\Gamma(1), \pi_\nu^{-\omega} \otimes \eta)$ of η-automorphic hyperfunctions from Section 3.

Proposition 4.2 (cf. [2], Prop. 2.1) *There is a bijective linear map*

$$\begin{array}{rcl} A_{\nu,\eta}^{-\omega} & \to & \mathcal{F}_{\nu,\eta} \\ \alpha & \mapsto & f_\alpha \end{array}$$

such that the function $z \mapsto (1+z^2)^{\nu+\frac{1}{2}} f_\alpha(z)$ *represents the restriction* $\alpha|_{\mathbb{R}}$.

Proof: The space $A_{\nu,\eta}^{-\omega} = (\pi_\nu^{-\omega} \otimes \eta)^{\Gamma(1)}$ can be viewed as the space of all V_η-valued hyperfunctions α in $\mathbb{P}_1(\mathbb{R})$ satisfying the invariance condition

$$\pi_\nu(\gamma^{-1})\alpha = \eta(\gamma)\alpha$$

for every $\gamma \in \Gamma(1)$. Pick a representative f for α. The V_η-valued function $F\colon z \mapsto (1+z^2)^{-\nu-\frac{1}{2}} f(z)$ is holomorphic on $0 < |\mathrm{Im}(z)| < \varepsilon$ for some $\varepsilon > 0$. Note that the invariance of α under T implies that for some function q, holomorphic on a neighbourhood of $\mathbb{R}$, we have

$$\begin{aligned} \eta(T)f(z) + q(z) &= \Big(\pi_\nu(T^{-1})f\Big)(z) \\ &= \left(\frac{1+z^2}{1+(z+1)^2}\right)^{\nu+\frac{1}{2}} f(z+1) \\ &= (1+z^2)^{\nu+\frac{1}{2}} F(z+1), \end{aligned}$$

so that

$$F(z+1) = \eta(T)F(z) + (1+z^2)^{-(\nu+\frac{1}{2})} q(z).$$

Therefore F represents a hyperfunction H on $\mathbb{P}_1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ which satisfies $H(z+1) = \eta(T)H(z)$.

Lemma 4.3 *Every hyperfunction on* $\mathbb{P}^1(\mathbb{R})$ *has a representative which is holomorphic in* $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$. *This representative is unique up to an additive constant.*

Proof: This is Lemma 2.3 in [4]. $\square$

By the lemma, the hyperfunction represented by F has a representative which is holomorphic in $\mathbb{P}_1(\mathbb{C}) \setminus \mathbb{P}_1(\mathbb{R})$. The freedom in this representative

is an additive constant. So there is a unique representative f_α of the form

$$f_\alpha(z) \;=\; \begin{cases} \dfrac{1}{2}v_0 + \displaystyle\sum_{k=1}^{\infty} e^{2\pi i \frac{k}{N} z} v_k^+, & z \in \mathbb{H}, \\ -\dfrac{1}{2}v_0 - \displaystyle\sum_{k=1}^{\infty} e^{-2\pi i \frac{k}{N} z} v_k^-, & z \in \bar{\mathbb{H}}. \end{cases}$$

So $f_\alpha \in \mathcal{F}_\eta$ and $(1+z^2)^{\nu+\frac{1}{2}} f_\alpha(z)$ represents $\alpha|_{\mathbb{R}}$. To show the injectivity of the map in the Proposition assume that $f_\alpha = 0$. Then α is supported in $\{\infty\}$. Since latter set is not Γ-invariant, α must be zero. To see that f_α lies in $\mathcal{F}_{\nu,\eta}$, recall that the invariance of α under S implies that

$$(1+z^{-2})^{\nu+\frac{1}{2}} f_\alpha\left(-\frac{1}{z}\right) \;=\; (1+z^2)^{\nu+\frac{1}{2}} \eta(S) f_\alpha\,(z) + \tilde{q}(z)$$

with $\tilde{q}(z)$ holomorphic on a neighbourhood of $\mathbb{R} \smallsetminus \{0\}$. Thus Proposition 4.1 shows that $f_\alpha \in \mathcal{F}_{\nu,\eta}$. To finally show surjectivity, let $f \in \mathcal{F}_{\nu,\eta}$. Then the function

$$z \mapsto (1+z^2)^{\nu+\frac{1}{2}} f(z)$$

represents a hyperfunction β_0 on $\mathbb{R}$ that satisfies $\pi_\nu(T^{-1})\beta_0 = \eta(T)\beta_0$. Let $\beta_\infty = (\pi_\nu \otimes \eta)(S)\beta_0$. Then β_∞ is a hyperfunction on $\mathbb{P}_1(\mathbb{R}) \smallsetminus \{0\}$ with representative $z \mapsto (1+z^{-2})^{\nu+\frac{1}{2}} \eta(S) f(\frac{-1}{z})$. According to Proposition 4.1 the restrictions of β_0 and β_∞ to $\mathbb{P}_1(\mathbb{R}) \smallsetminus \{0,\infty\}$ agree. Thus β_0 and β_∞ are restrictions of a hyperfunction β on $\mathbb{P}_1(\mathbb{R})$ which satisfies $\pi_\nu \otimes \eta(S)\beta = \beta$. Using β_0 we see that the support of

$$(\pi_\nu \otimes \eta)(T)\beta - \beta$$

is contained in $\{\infty\}$. Using β_∞ we see that for $|z| > 2$, $z \notin \mathbb{R}$, this hyper-

function is represented by

$$\begin{aligned}
&\left(\frac{1+z^2}{1+(z-1)^2}\right)^{\nu+\frac{1}{2}}\left(1+(z-1)^{-2}\right)^{\nu+\frac{1}{2}}\eta(TS)f\left(\frac{-1}{z-1}\right)\\
&\qquad -(1+z^{-2})^{\nu+\frac{1}{2}}\eta(S)f\left(\frac{-1}{z}\right)=\\
=\;& (1+z^2)^{\nu+\frac{1}{2}}(z-1)^{-2\nu-1}\eta(TS)f\left(\frac{-1}{z-1}\right)\\
&\qquad -(1+z^{-2})^{\nu+\frac{1}{2}}\eta(S)f\left(\frac{-1}{z}\right)\\
=\;& (1+z^{-2})^{\nu+\frac{1}{2}}\times\\
&\left(\left(\frac{z}{z-1}\right)^{2\nu+1}\eta(TST^{-1})f\left(\frac{z-2}{z-1}\right)-\eta\left(ST^{-1}\right)f\left(\frac{z-1}{z}\right)\right)
\end{aligned}$$

Since $f(z)$ is holomorphic around $z=1$ it follows that this function is holomorphic around $z=\infty$. Hence β is invariant under T. Now the claim follows because the elements S and T generate $\Gamma(1)$. $\square$

Proposition 4.4 (Bruggeman transform; cf. [2], Prop. 2.3) *For* $\alpha\in A^{-\omega}_{\nu,\eta}$ *put*

$$\boxed{\psi_\alpha(z)} = f_\alpha(z)-z^{-2\nu-1}\eta(S)f_\alpha\left(\frac{-1}{z}\right),$$

with f_α *as in Proposition 4.2. Then the* Bruggeman transform $B\colon\alpha\mapsto\psi_\alpha$ *maps* $A^{-\omega}_{\nu,\eta}$ *to* $\Psi_{\nu,\eta}$. *It is a bijection if* $\nu\notin\frac{1}{2}+\mathbb{Z}$.

Proof: Let $\alpha\in A^{-\omega}_{\nu,\eta}$ and define ψ_α as in the Proposition. By Proposition 4.2 the map ψ_α extends to $\mathbb{C}\setminus(-\infty,0]$. We compute

$$\begin{aligned}
&\psi_\alpha(z+1)+(z+1)^{-2\nu-1}\eta(ST^{-1})\psi_\alpha\left(\frac{z}{z+1}\right)=\\
=\;& f_\alpha(z+1)-(z+1)^{-2\nu-1}\eta(S)f_\alpha\left(\frac{-1}{z+1}\right)+(z+1)^{-2\nu-1}\eta(ST^{-1})\times\\
&\times\left(f_\alpha\left(\frac{z}{z+1}\right)-\left(\frac{z}{z+1}\right)^{-2\nu-1}\eta(S)f_\alpha\left(\frac{-1}{\frac{z}{z+1}}\right)\right).
\end{aligned}$$

Since $\frac{z}{z+1} = 1 - \frac{1}{z+1}$ and $f_\alpha\left(1 - \frac{1}{z+1}\right) = \eta(T) f_\alpha\left(\frac{-1}{z+1}\right)$ we see that the two middle summands cancel out. It remains

$$\begin{aligned}
&\eta(T) f_\alpha(z) - z^{-2\nu-1}\eta(ST^{-1}S) f_\alpha\left(\frac{-z-1}{z}\right) = \\
&= \eta(T) f_\alpha(z) - z^{-2\nu-1}\eta(ST^{-1}ST^{-1}) f_\alpha\left(\frac{-1}{z}\right) \\
&= \eta(T)\left(f_\alpha(z) - z^{-2\nu-1}\eta(S) f_\alpha\left(\frac{-1}{z}\right)\right) \\
&= \eta(T)\psi_\alpha(z).
\end{aligned}$$

Here we have used $ST^{-1}ST^{-1} = TS$. This proves that ψ_α satisfies the Lewis equation (1).

Next, if $\nu \in \frac{1}{2} + \mathbb{Z}$, then one sees from the definition of ψ_α that $\psi_\alpha(z) + z^{-2\nu-1}\eta(S)\psi_\alpha\left(\frac{-1}{z}\right)$ equals zero and so ψ_α lies in $\Psi_{\nu,\eta}$. If $\nu \notin \frac{1}{2} + \mathbb{Z}$ then recall that we take the standard branch of the logarithm to define $z^{-2\nu-1}$. One gets the inversion formula

$$f_\alpha(z) = \frac{1}{1 + e^{\mp 2\pi i\nu}}\left(\psi_\alpha(z) + z^{-2\nu-1}\eta(S)\psi_\alpha\left(\frac{-1}{z}\right)\right) \tag{7}$$

for $z \in \mathbb{H}^\pm$. Since $f_\alpha \in \mathcal{F}_{\nu,\eta} \subseteq \mathcal{F}_\eta$, it satisfies $f_\alpha(i\infty) + f_\alpha(-i\infty) = 0$ and this implies

$$\begin{aligned}
0 &= \frac{1}{1+e^{-2\pi i\nu}}\left(\lim_{\mathrm{Im} z\to\infty}\left(\psi_\alpha(z) + z^{-2\nu-1}\eta(S)\psi_\alpha\left(\tfrac{-1}{z}\right)\right)\right) \\
&\quad + \frac{1}{1+e^{+2\pi i\nu}}\left(\lim_{\mathrm{Im} z\to-\infty}\left(\psi_\alpha(z) + z^{-2\nu-1}\eta(S)\psi_\alpha\left(\tfrac{-1}{z}\right)\right)\right) \\
&= \frac{e^{-\pi i\nu}}{1+e^{-2\pi i\nu}}\left(e^{\pi i\nu}\lim_{\mathrm{Im} z\to\infty}\left(\psi_\alpha(z) + z^{-2\nu-1}\eta(S)\psi_\alpha\left(\tfrac{-1}{z}\right)\right)\right) \\
&\quad + \frac{e^{\pi i\nu}}{1+e^{+2\pi i\nu}}\left(e^{-\pi i\nu}\lim_{\mathrm{Im} z\to-\infty}\left(\psi_\alpha(z) + z^{-2\nu-1}\eta(S)\psi_\alpha\left(\tfrac{-1}{z}\right)\right)\right) \\
&= \frac{e^{-\pi i\nu}}{1+e^{-2\pi i\nu}}\left(e^{\pi i\nu}\lim_{\mathrm{Im} z\to\infty}\left(\psi_\alpha(z) + z^{-2\nu-1}\eta(S)\psi_\alpha\left(\tfrac{-1}{z}\right)\right)\right. \\
&\quad \left. + e^{-\pi i\nu}\lim_{\mathrm{Im} z\to-\infty}\left(\psi_\alpha(z) + z^{-2\nu-1}\eta(S)\psi_\alpha\left(\tfrac{-1}{z}\right)\right)\right).
\end{aligned}$$

This proves $B\alpha = \psi_\alpha \in \Psi_{\nu,\eta}$ and it only remains to show that the Bruggeman transform is surjective. But a simple calculation, similar to the one

given above shows that for a holomorphic function $\psi\colon \mathbb{C}\smallsetminus(-\infty,0] \to V_\eta$ satisfying the Lewis equation (1), the function $f\colon \mathbb{C}\smallsetminus\mathbb{R}\to V_\eta$, defined from ψ via the inversion formula (7), satisfies $f(z+1)=\eta(T)f(z)$. If ψ satisfies (2), then the above calculation shows that f is bounded as $|\mathrm{Im}(z)|\to\infty$ and $f(i\infty)+f(-i\infty)=0$. In view of Proposition 4.2 this, finally, proves the claim. $\square$

5 Maaß wave forms

Recall the space $\mathcal{S}_{\nu,q,\eta}$ of *Maaß cusp forms* from Section 2 and consider a function u in $\mathcal{S}_{\nu,q,\eta}$. Because of $u(z+N)=u(z)$ the function u has a Fourier series expansion

$$u(z) \;=\; u(x+iy) \;=\; \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} e^{2\pi i \frac{k}{N}x}\, v_k(y)$$

for some smooth function $v_k:(0,\infty)\to V_\eta$. The differential equation $\Delta u = (\frac{1}{4}-\nu^2)u$ induces a differential equation for $v_k(y)$ which implies that its coordinates must linear combinations of I and K-Bessel functions. The fact that u is square integrable rules out the I-Bessel functions, so we can assume

$$v_k(y) \;=\; \sqrt{y}K_\nu\left(2\pi\frac{|k|}{N}y\right)v_k$$

for some $v_k\in V_\eta$. By Theorem 3.2 of [13] it follows that the norms $\|v_k\|$ are bounded as $|k|\to\infty$. The functional equation $u(z+1)=\eta(T)u(z)$ is reflected in the fact that the v_k are eigenvectors of $\eta(T)$, since it implies $\eta(T)v_k = e^{2\pi i\frac{k}{N}}v_k$. Now set

$$f_u(z) \;=\; \begin{cases} \displaystyle\sum_{k>0} k^\nu e^{2\pi i\frac{k}{N}z}v_k, & \mathrm{Im}(z)>0,\\ \displaystyle -\sum_{k<0} |k|^\nu e^{2\pi i\frac{k}{N}z}v_k, & \mathrm{Im}(z)<0. \end{cases}$$

From the construction it is clear that $f_u\in\mathcal{F}_\eta$. It will play the role of our earlier f_α (cf. Proposition 4.2), so we define

$$\psi_u(z) \;=\; f_u(z) - z^{-2\nu-1}\eta(S)f_u\left(\frac{-1}{z}\right).$$

Lemma 5.1 *For* $\mathrm{Re}(\nu) > -\frac{1}{2}$ *the equations above define linear maps*

$$\begin{array}{rcl} \mathcal{S}_{\nu,q,\eta} & \to & \mathcal{F}_{\nu,\eta} \\ u & \mapsto & f_u \end{array} \quad \textit{and} \quad \begin{array}{rcl} \mathcal{S}_{\nu,q,\eta} & \to & \Psi^o_{\nu,\eta}. \\ u & \mapsto & \psi_u \end{array}$$

Proof: This is a rather straightforward extension of the proofs in [16] Chap I. We contain it here for the convenience of the reader. To prove that $f_u \in \mathcal{F}_{\nu,\eta}$ we will need the following two Dirichlet series. For $\varepsilon = 0, 1$ set

$$L_\varepsilon(u,s) \;=\; \sum_{k\neq 0} \mathrm{sign}(k)^\varepsilon \left(\frac{N}{|k|}\right)^s v_k.$$

We will relate L_0 and L_1 to u by the Mellin transform. For this let

$$u_0(y) \;=\; \frac{1}{\sqrt{y}} u(iy), \qquad u_1(y) \;=\; \frac{\sqrt{y}}{2\pi i} u_x(iy),$$

where $u_x = \frac{\partial}{\partial x} u$. Next define

$$\hat{L}_\varepsilon(u,s) \;=\; \int_0^\infty u_\varepsilon(y) y^s \frac{dy}{y}.$$

Plugging in the Fourier series of u and using the fact that

$$\int_0^\infty K_\nu(2\pi y) y^s \frac{dy}{y} \;=\; \Gamma_\nu(s) \;\stackrel{\mathrm{def}}{=}\; \frac{1}{4\pi^s} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu}{2}\right),$$

we get

$$\hat{L}_0(u,s) \;=\; \Gamma_\nu(s) L_0(u,s),$$

and similarly,

$$\hat{L}_1(u,s) \;=\; \Gamma_\nu(s+1) L_1(u,s).$$

On the other hand, the usual process of splitting the Mellin integral and using the functional equations

$$u_\varepsilon\left(\frac{1}{y}\right) \;=\; (-1)^\varepsilon y\, \eta(S)\, u_\varepsilon(y), \qquad \varepsilon = 0, 1,$$

(which are implied by $u(\gamma z) = \eta(\gamma) u(z)$ for $\gamma \in \Gamma$), one gets that $\hat{L}_\varepsilon$ extends to an entire function and satisfies the functional equation,

$$\hat{L}_\varepsilon(u,s) \;=\; (-1)^\varepsilon\, \eta(S)\, \hat{L}_\varepsilon(u, 1-s).$$

With a similar, even easier computation one gets

$$\int_0^\infty y^s \left(f_u(iy) - (-1)^\varepsilon f_u(-iy)\right) \frac{dy}{y} \;=\; \frac{\Gamma(s)N^\nu}{(2\pi)^s} L_\varepsilon(u, s-\nu).$$

This implies that the *Mellin transforms* $M^\pm f_u(s) = \int_0^\infty y^s f_u(\pm iy)\,\frac{dy}{y}$ can be calculated as

$$\begin{aligned} M^\pm f_u(s) &= \pm\tfrac{\Gamma(s)N^\nu}{2(2\pi)^s}\left(L_0(u,s-\nu) \pm L_1(u,s-\nu)\right) \\ &= \pm N^\nu \pi^{-\nu-\frac{3}{2}}\Gamma\left(\tfrac{s+1}{2}\right)\Gamma\left(\tfrac{2\nu+2-s}{2}\right)\sin\pi\left(\nu+1-\tfrac{s}{2}\right)\hat{L}_0(u,s-\nu) \\ &\quad + N^\nu \pi^{-\nu-\frac{1}{2}}\Gamma\left(\tfrac{s}{2}\right)\Gamma\left(\tfrac{2\nu+1-s}{2}\right)\sin\pi\left(\nu+\tfrac{1}{2}-\tfrac{s}{2}\right)\hat{L}_1(u,s-\nu). \end{aligned}$$

The last identity follows from the standard equations

$$\Gamma\left(\tfrac{x}{2}\right)\Gamma\left(\tfrac{x+1}{2}\right) \;=\; \Gamma(x)2^{1-x}\sqrt{\pi}, \qquad \Gamma(x)\Gamma(1-x) \;=\; \tfrac{\pi}{\sin\pi x}.$$

Thus the Mellin transform $M^\pm f_u(s)$ is seen to be holomorphic for $\mathrm{Re}\,(s) > 0$ and rapidly decreasing on any vertical strip. The Mellin inversion formula yields for $C > 0$,

$$f_u(\pm iy) \;=\; \frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=C} y^{-s} M^\pm f_u(s)\,ds, \quad y > 0.$$

This extends to any $z \in \mathbb{C} \smallsetminus \mathbb{R}$ to give

$$f_u(z) \;=\; \frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=C} e^{\pm\frac{\pi}{2}is} z^{-s}\, M^\pm f_u(s)\,ds$$

for $z \in \mathbb{H}^\pm$. For $C > 0$ it follows that

$$\psi_u(z) \;=\frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=C}\left(e^{\pm\frac{\pi}{2}is}z^{-s} - e^{\mp\frac{\pi}{2}is}z^{-2\nu-1}z^s\eta(S)\right) M^\pm f_u(s)\,ds.$$

Writing this as the difference of two integrals, substituting s in the second integral with $2\nu+1-s$ and shifting the contour, for $0 < C < 2\mathrm{Re}\,(\nu)+1$ we arrive at the formula

$$\tfrac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=C} \begin{array}{l}\Big(e^{\pm\frac{\pi}{2}is}z^{-s}M^\pm f_u(s) \\ \quad -e^{\mp\frac{\pi}{2}i(2\nu+1-s)}z^{-s}\eta(S)M^\pm f_u(2\nu+1-s)\Big)\,ds.\end{array} \tag{8}$$

for ψ_u. Using the identities

$$\begin{aligned} \pm e^{\pm\frac{\pi}{2}is}\cos\pi\left(\nu+\frac{1}{2}-\frac{s}{2}\right)\mp e^{\mp\frac{\pi}{2}i(2\nu+1-s)}\cos\pi\frac{s}{2} &= i\sin\pi\left(\nu+\frac{1}{2}\right), \\ e^{\pm\frac{\pi}{2}is}\sin\pi\left(\nu+\frac{1}{2}-\frac{s}{2}\right)+e^{\mp\frac{\pi}{2}i(2\nu+1-s)}\sin\pi\frac{s}{2} &= \sin\pi\left(\nu+\frac{1}{2}\right), \end{aligned}$$

and the functional equation of $\hat{L}_\varepsilon$ we see that the integrand of (8) equals

$$\begin{aligned} z^{-s}N^\nu\sin\pi\left(\nu+\tfrac{1}{2}\right)\quad &\Big[\pi^{-\nu-\frac{3}{2}}\Gamma\left(\tfrac{s+1}{2}\right)\Gamma\left(\tfrac{2\nu+2-s}{2}\right)i\,\hat{L}_0(u,s-\nu)+ \\ &+\pi^{-\nu-\frac{1}{2}}\Gamma\left(\tfrac{s}{2}\right)\Gamma\left(\tfrac{2\nu+1-s}{2}\right)\,\hat{L}_1(u,s-\nu)\Big]. \end{aligned} \tag{9}$$

Since this expression is independent of whether z lies in $\mathbb{H}$ or $\bar{\mathbb{H}}$, it follows that the function $f_u(z)-z^{-2\nu-1}\eta(S)f_u\left(\frac{-1}{z}\right)$ extends to a holomorphic function on $\mathbb{C}\setminus(-\infty,0]$, i.e., the function f_u indeed lies in the space $\mathcal{F}_{\nu,\eta}$. The linearity of the map is clear.

It remains to show that $\psi_u\in\Psi^o_{\nu,\eta}$. Note that in view of $f_u\in\mathcal{F}_{\nu,\eta}$ Proposition 4.2 shows that the function $z\mapsto(1+z^2)^{\nu+\frac{1}{2}}f_u(z)$ represents the restriction $\alpha_u|_\mathbb{R}$ of a hyperfunction $\alpha_u\in A^{-\omega}_{\nu,\eta}$. Then, according to Proposition 4.4 we have $\psi_u=B(\alpha_u)$ so ψ_u satisfies (2). The asymptotic property (3) now follows from the integral representation (8) with the C chosen there. More precisely, the bound $O(|z|^{-C})$ follows directly from (9) since the integrant divided by z^{-s} is of π-exponential decay, whereas the $O(1)$-bound is obtained by moving the contour slightly to the left of the imaginary axis picking up the residue at 0 which is proportional to 1 (see [16], §I.4 for more details on this type of argument). □

Note that according to [2], §3 the canonical pairing $\pi^\omega_{-\nu}\times\pi^{-\omega}_{-\nu}\to\mathbb{C}$ and hence also the pairing $\pi^\omega_{-\nu}\times\pi^{-\omega}_{-\nu}\otimes\eta\to V_\eta$ can be calculated in terms of path integrals once we realize the principal series on $L^2(\mathbb{R},\frac{1}{\pi}\frac{dx}{1+x^2})$.

Lemma 5.2 *For $0\neq k\in\mathbb{Z}$ let α_k be the hyperfunction on $\mathbb{P}_1(\mathbb{R})$ the restriction to $\mathbb{R}$ of which is represented by $(1+z^2)^{\nu+\frac{1}{2}}f_k(z)$ with*

$$f_k(z)=\begin{cases}\operatorname{sign}(k)\cdot e^{2\pi i\frac{k}{N}z} & \textit{for}\ \operatorname{sign}(k)\cdot\operatorname{Im}(z)>0\\ 0 & \textit{for}\ \operatorname{sign}(k)\cdot\operatorname{Im}(z)<0.\end{cases}$$

Then we have that

$$\left\langle\pi_{-\nu}\begin{pmatrix}\sqrt{b} & \frac{a}{\sqrt{b}}\\ 0 & \frac{1}{\sqrt{b}}\end{pmatrix}\varphi_0,\alpha_k\right\rangle$$

equals

$$2\,\mathrm{sign}(k)\left(\frac{N}{|k|}\right)^{\nu}\frac{\pi^{-\nu-\frac{1}{2}}}{\Gamma(\frac{1}{2}-\nu)}\sqrt{b}\;K_\nu\left(2\pi\frac{|k|}{N}b\right)e^{2\pi i\frac{k}{N}a},$$

where is K_ν the K-Bessel function with parameter ν.

Proof: For this we will need the following identity (cf. [2], §4 or [18], p.136), which holds for $y>0$,

$$\int_{-\infty}^{\infty}\left(\frac{1}{y^2+(\tau-x)^2}\right)^{\frac{1}{2}-\nu}e^{2\pi ik\tau}\,d\tau=\frac{2\pi^{\frac{1}{2}-\nu}|k|^{-\nu}}{\Gamma\left(\frac{1}{2}-\nu\right)}y^\nu K_\nu(2\pi|k|y)e^{2\pi ikx}.$$

Note that $g=\begin{pmatrix}\sqrt{b} & \frac{a}{\sqrt{b}}\\ 0 & \frac{1}{\sqrt{b}}\end{pmatrix}$ satisfies $g\cdot i=a+ib$. Therefore, by abuse of notation, we write $P(\alpha_k)(a+ib)$ for $\left\langle\pi_{-\nu}\begin{pmatrix}\sqrt{b} & \frac{a}{\sqrt{b}}\\ 0 & \frac{1}{\sqrt{b}}\end{pmatrix}\varphi_0,\alpha_k\right\rangle$. According to [2], §4, we can calculate

$$\begin{aligned}P(\alpha_k)(a+ib) &= \left\langle\left(\frac{1+x^2}{b+\left(\frac{x}{\sqrt{b}}-\frac{a}{\sqrt{b}}\right)^2}\right)^{-\nu+\frac{1}{2}},\alpha_k\right\rangle\\ &= b^{-\nu+\frac{1}{2}}\left\langle\left(\frac{1+x^2}{b^2+(x-a)^2}\right)^{-\nu+\frac{1}{2}},(1+z^2)^{\nu+\frac{1}{2}}f_k(z)\right\rangle\\ &= \mathrm{sign}(k)b^{-\nu+\frac{1}{2}}\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{b^2+(x-a)^2}\right)^{-\nu+\frac{1}{2}}e^{2\pi i\frac{k}{N}x}\,dx\\ &= 2\,\mathrm{sign}(k)\left(\frac{N}{|k|}\right)^{\nu}\frac{\pi^{-\nu-\frac{1}{2}}}{\Gamma(\frac{1}{2}-\nu)}\sqrt{b}\;K_\nu\left(2\pi\frac{|k|}{N}b\right)e^{2\pi i\frac{k}{N}a},\end{aligned}$$

where in the last step we have used the integral representation of the K-Bessel function above. □

Theorem 5.3 (Lewis transform; cf. [16], Thm. 1.1) *For $\nu\notin\frac{1}{2}+\mathbb{Z}$ and $\mathrm{Re}(\nu)>-\frac{1}{2}$ the Lewis transform is a bijective linear map from the space of Maaß cusp forms $\mathcal{S}_{\nu,q,\eta}$ to the space $\Psi^o_{\nu,\eta}$ of period functions.*

Proof: Since $\nu\notin\frac{1}{2}+\mathbb{Z}$ and $L=B\circ P^{-1}$, Proposition 4.4 shows that $L\colon\mathcal{S}_{\nu,q,\eta}\to\Psi_{\nu,\eta}$ is injective.

It remains to show that $L(\mathcal{S}_{\nu,\eta}) = \Psi^o_{\nu,\eta}$. To do this pick $\psi \in \Psi^o_{\nu,\eta}$. According to Propositions 4.2 and 4.4 we can find a hyperfunction $\alpha \in A^{-\omega}_{\nu,\eta}$ represented by the function $(1+z^2)^{\nu+\frac{1}{2}} f$ with $f \in \mathcal{F}_{\nu,\eta}$ such that

$$\begin{aligned} \psi(z) &= f(z) - z^{-2\nu-1}\eta(S) f\left(-\frac{1}{z}\right), \\ f(z) &= \frac{1}{1+e^{\mp 2\pi i\nu}}\left(\psi(z) + z^{-2\nu-1}\eta(S)\psi\left(\frac{-1}{z}\right)\right) \end{aligned}$$

for $z \in \mathbb{H}^{\pm}$. The function f admits a Fourier expansion of the form

$$f(z) = \begin{cases} \frac{1}{2}v_0 + \sum_{k=1}^{\infty} e^{2\pi i \frac{k}{N} z} v_k, & z \in \mathbb{H}, \\ -\frac{1}{2}v_0 - \sum_{k=1}^{\infty} e^{-2\pi i \frac{k}{N} z} v_{-k}, & z \in \bar{\mathbb{H}}. \end{cases} \tag{10}$$

The asymptotic property (3) of ψ implies that

$$\begin{aligned} \psi(z) &= O(|z|^{-C}) \\ z^{-2\nu-1}\eta(S)\psi\left(-\tfrac{1}{z}\right) &= O(|z|^{-2\mathrm{Re}\,(\nu)-1}) \end{aligned}$$

for $z \in \mathbb{C} \setminus (-\infty, 0]$. Since $2\mathrm{Re}\,(\nu) + 1 > 0$ this implies that there is a constant $\epsilon > 0$ such that

$$f(x+iy) = O(|y|^{-\varepsilon})$$

locally uniformly in x. Since f is periodic, this shows $v_0 = 0$. Note that $K_\nu(t) \sim e^{-t}\sqrt{\frac{\pi}{2t}}$ for $t \to \infty$(see [18] p. 137). Therefore we have

$$A_k(y) = \sqrt{y} K_\nu\left(2\pi \frac{|k|}{N} y\right) \sim e^{-2\pi\frac{|k|}{N}y}\sqrt{\frac{N}{4\,|k|}}$$

uniformly in k, which implies that

$$u(z) = u(x+iy) = \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \frac{\mathrm{sign}(k)}{|k|^\nu} A_k(y)\, e^{2\pi i \frac{k}{N} x}\, v_k$$

defines a smooth function on $\mathbb{H}$. Taking the derivatives termwise, we see that u is an eigenfunction of the Laplacian.

Using the V_η-valued pairing $V^{\omega}_{\pi_\nu} \otimes \mathcal{A}^{-\omega}_{\nu,\eta} \to V_\eta$ induced by the natural pairing $V^{\omega}_{\pi_\nu} \otimes V^{-\omega}_{\pi_\nu} \to \mathbb{C}$ we conclude from Lemma 5.2 that

$$\left\langle \pi_{-\nu} \begin{pmatrix} \sqrt{b} & \frac{a}{\sqrt{b}} \\ 0 & \frac{1}{\sqrt{b}} \end{pmatrix} \varphi_0, \alpha \right\rangle \;=\; Cu(a+bi)$$

for a constant $C \neq 0$ which depends on ν. Therefore u satisfies $\int_0^N u(z+t)\,dt = 0$ for every $z \in \mathbb{H}$, and so $u \in \tilde{\mathcal{M}}_{\nu,\eta}$. Note that this equation is a consequence of $v_0 = 0$. Thus in order to show that $\psi \in L(\mathcal{S}_{\nu,q,\eta})$, it only remains to show that u is square integrable. But the asymptotic of A_k above implies that u rapidly decreases towards the cusp and hence the finite volume of the fundamental domain proves the square integrability of u. Thus we have shown that $L(\mathcal{S}_{\nu,\eta}) \supseteq \Psi^o_{\nu,\eta}$. For the converse we note that any $u \in \mathcal{S}_{\nu,\eta}$ can be written as a Fourier series, so that $f_u(z)$ is given by (10). Then the above calculation shows that there exists a nonzero constant C depending on ν such that $P\alpha_u = Cu$. Thus $Lu = B \circ P^{-1}u = \frac{1}{C}B\alpha_u = \frac{1}{C}\psi_u \in \Psi^o_{\nu,\eta}$ by Lemma 5.1. □

As a consequence of this proof we see that for η the trivial representation, our Lewis transform coincides with $\frac{1}{2}\pi^{\nu+\frac{1}{2}}\Gamma\left(\frac{1}{2}-\nu\right)$ times the one given in [16].

6 Characterizing period functions on $\mathbb{R}^+$

Let $T' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \left(STS^{-1}\right)^{-1} = TS^{-1}T$ and note that $S = S^{-1}$. Then $\eta(T)$ and $\eta(T')$ have the same order N. Note that the operators $\eta(T)$ and $\eta(T')$ are of finite order, so the sequences $\eta(T(T')^n)$ and $\eta(T'T^n)$ are bounded. Consider the operator valued Hurwitz type zeta functions

$$\zeta_\eta(a,x) = \sum_{n=0}^{\infty} \frac{\eta(T(T')^n)^{-1}}{(n+x)^a} \quad \text{and} \quad \zeta'_\eta(a,x) = \sum_{n=0}^{\infty} \frac{\eta(T'T^n)^{-1}}{(n+x)^a}.$$

Then

$$\begin{aligned}
\zeta_\eta(a,x) &= \frac{1}{N^a} \sum_{j=0}^{N-1} \eta(T(T')^j)^{-1} \zeta\big(a, \tfrac{j+x}{N}\big), \\
\zeta'_\eta(a,x) &= \frac{1}{N^a} \sum_{j=0}^{N-1} \eta(T'T^j)^{-1} \zeta\big(a, \tfrac{j+x}{N}\big),
\end{aligned}$$

where $\zeta(a,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^a}$ is the ordinary Hurwitz zeta function. The Hurwitz zeta function satisfies

$$\zeta(a,x) \underset{x\to\infty}{\sim} \frac{1}{a-1} \sum_{k\geq 0} (-1)^k B_k \binom{k+a-2}{k} x^{-a-k+1}$$

for the Bernoulli numbers B_k. For ζ_η and ζ'_η this results in

$$\zeta_\eta(a,x) \underset{x\to\infty}{\sim} \frac{1}{a-1} \sum_{\mu=0}^{\infty} x^{1-a-\mu} \sum_{j=0}^{N-1} \eta(T(T')^j)^{-1} C(\mu,j) \tag{11}$$

and

$$\zeta'_\eta(a,x) \underset{x\to\infty}{\sim} \frac{1}{a-1} \sum_{\mu=0}^{\infty} x^{1-a-\mu} \sum_{j=0}^{N-1} \eta(T'T^j)^{-1} C(\mu,j), \tag{12}$$

where

$$C(\mu,j) = \sum_{k=0}^{\mu} (-1)^k B_k \binom{k+a-2}{k} \binom{1-a-k}{\mu-k} N^{k-1} j^{\mu-k}.$$

Lemma 6.1 (cf. [16], §III.3) *If a smooth function $\psi\colon (0,\infty) \to V_\eta$ satisfies* (4) *with $\nu \notin \frac{1}{2}+\mathbb{Z}$, then it has the following asymptotic expansions:*

$$\begin{aligned}
\psi(x) &\underset{x\to 0}{\sim} x^{-2\nu-1} Q_0\big(\tfrac{1}{x}\big) + \sum_{l=-1}^{\infty} C_l^* x^l, \\
\psi(x) &\underset{x\to\infty}{\sim} Q_\infty(x) + \sum_{l=-1}^{\infty} C_l^{*\prime} x^{-l-2\nu-1},
\end{aligned}$$

where the $Q_0, Q_\infty\colon R \to V_\eta$ are smooth functions with

$$\begin{aligned}
Q_0(x+1) &= \eta(T') Q_0(x), \\
Q_\infty(x+1) &= \eta(T) Q_\infty(x),
\end{aligned}$$

and the C_l^* and the $C_l^{*\prime}$ can be calculated from the Taylor coefficients $\boxed{C_m} = \frac{1}{m!}\psi^{(m)}(1) \in V_\eta$ of ψ in 1 via

$$C_l^* = \sum_{m=0}^{l+1} \frac{1}{m+2\nu} \sum_{j=0}^{N-1} \eta(T(T')^j)^{-1} C_m \, C(l+1-m,j),$$

and

$$C_l^{*'} = \sum_{m=0}^{l+1} \frac{1}{m+2\nu} \sum_{r=0}^{l+1-m} \binom{r-2\nu-l-1}{r} \sum_{j=0}^{N-1} \eta(T'T^j)^{-1} C_m C(l{+}1{-}r{-}m, j).$$

If ψ is real analytic, then so are Q_0 and Q_∞.

Proof: For $\mathrm{Re}(\nu) > 0$ set

$$Q_0(x) = x^{-2\nu-1}\psi\left(\tfrac{1}{x}\right) - \sum_{n=0}^{\infty}(n+x)^{-2\nu-1}\eta\big(T(T')^n\big)^{-1}\psi\left(1+\frac{1}{n+x}\right)$$

and

$$Q_\infty(x) = \psi(x) - \sum_{n=1}^{\infty}(n+x)^{-2\nu-1}\eta\big(T'T^{n-1}\big)^{-1}\psi\left(1-\frac{1}{n+x}\right).$$

Then we have

$$\begin{aligned}
&Q_0(x+1) - \eta(T')Q_0(x) = \\
&= (x+1)^{-2\nu-1}\psi\left(\frac{1}{x+1}\right) - x^{-2\nu-1}\eta(T')\psi\left(\frac{1}{x}\right) \\
&\quad - \sum_{n=0}^{\infty}(n+1+x)^{-2\nu-1}\eta(T(T')^n)^{-1}\psi\left(1+\frac{1}{n+1+x}\right) \\
&\quad + \sum_{n=0}^{\infty}(n+x)^{-2\nu-1}\eta(T')\eta(T(T')^n)^{-1}\psi\left(1+\frac{1}{n+x}\right) \\
&= (x+1)^{-2\nu-1}\psi\left(\frac{1}{x+1}\right) - x^{-2\nu-1}\eta(T')\psi\left(\frac{1}{x}\right) \\
&\quad - \sum_{n=1}^{\infty}(n+x)^{-2\nu-1}\eta(T(T')^{n-1})^{-1}\psi\left(1+\frac{1}{n+x}\right) \\
&\quad + \sum_{n=0}^{\infty}(n+x)^{-2\nu-1}\eta(T(T')^{n-1})^{-1}\psi\left(1+\frac{1}{n+x}\right) \\
&= (x+1)^{-2\nu-1}\psi\left(\frac{1}{x+1}\right) - x^{-2\nu-1}\eta(T')\psi\left(\frac{1}{x}\right) \\
&\quad + x^{-2\nu-1}\eta(T(T')^{-1})^{-1}\left(\eta(T)\psi\left(\frac{1}{x}\right)\right. \\
&\quad \left. - \left(1+\frac{1}{x}\right)^{-2\nu-1}\eta(ST^{-1})\psi\left(\frac{\frac{1}{x}}{1+\frac{1}{x}}\right)\right) \\
&= 0,
\end{aligned}$$

since $T^{-1}ST^{-1} = (T')^{-1}$. Similarly we calculate

$$
\begin{aligned}
&Q_\infty(x+1) - \eta(T)Q_\infty(x) = \\
&\quad = \psi(x+1) - \eta(T)\psi(x) \\
&\qquad - \sum_{n=1}^{\infty}(n+1+x)^{-2\nu-1}\eta(T'T^{n-1})^{-1}\psi\left(1-\frac{1}{n+1+x}\right) \\
&\qquad + \sum_{n=1}^{\infty}(n+x)^{-2\nu-1}\eta(T)\eta(T'T^{n-1})^{-1}\psi\left(1-\frac{1}{n+x}\right) \\
&\quad = \psi(x+1) - \eta(T)\psi(x) \\
&\qquad - \sum_{n=2}^{\infty}(n+x)^{-2\nu-1}\eta(T'T^{n-2})^{-1}\psi\left(1-\frac{1}{n+x}\right) \\
&\qquad + \sum_{n=1}^{\infty}(n+x)^{-2\nu-1}\eta(T'T^{n-2})^{-1}\psi\left(1-\frac{1}{n+x}\right) \\
&\quad = \psi(x+1) - \eta(T)\psi(x) \\
&\qquad +(1+x)^{-2\nu-1}\eta(T'T^{-1})^{-1}\psi\left(1-\frac{1}{x+1}\right) \\
&\quad = 0.
\end{aligned}
$$

For general ν and $-2\mathrm{Re}(\nu) - 1 < M \in \mathbb{N}$ we write

$$
\begin{aligned}
Q_0(x) \quad &\stackrel{\text{def}}{=} \quad x^{-2\nu-1}\psi\left(\tfrac{1}{x}\right) - \sum_{m=0}^{M}\zeta_\eta(m+2\nu+1,x)C_m \\
&\qquad - \sum_{n=0}^{\infty}(n+x)^{-2\nu-1}\eta\big(T(T')^n\big)^{-1}\left(\psi\left(1+\frac{1}{n+x}\right) - \sum_{m=0}^{M}\frac{C_m}{(n+x)^m}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
Q_\infty(x) \quad &\stackrel{\text{def}}{=} \quad \psi(x) - \sum_{m=0}^{M}\zeta'_\eta(m+2\nu+1,x+1)C_m \\
&\qquad - \sum_{n=1}^{\infty}(n+x)^{-2\nu-1}\eta\big(T'T^{n-1}\big)^{-1}\left(\psi\left(1-\frac{1}{n+x}\right) - \sum_{m=0}^{M}\frac{C_m}{(n+x)^m}\right),
\end{aligned}
$$

and note that the definition does not depend on M. Using $\zeta(a,x) = O(x^{1-a})$ we find

$$\psi(x) = x^{-2\nu-1} Q_0(x^{-1}) + \sum_{m=0}^{M} x^{-2\nu-1} \zeta_\eta(m+2\nu+1, x^{-1}) C_m$$
$$+ x^{-2\nu-1} \underbrace{\sum_{n=0}^{\infty} \frac{\eta\big(T(T')^n\big)^{-1}}{(x^{-1}+n)^{2\nu+1}} \left(\psi\left(1+\frac{1}{n+x^{-1}}\right) - \sum_{m=0}^{M} \frac{C_m}{(n+x^{-1})^m} \right)}_{=O(x^{2\nu+1+M})}.$$

Note that (11) implies

$$\sum_{m=0}^{M} x^{-2\nu-1} \zeta_\eta(m+2\nu+1, x^{-1}) C_m \underset{x\to 0}{\sim}$$
$$\underset{x\to 0}{\sim} \sum_{l=-1}^{\infty} \underbrace{\sum_{m=0}^{M} \frac{1}{m+2\nu} \sum_{j=0}^{N-1} \eta(T(T')^j)^{-1} C_m C(l+1-m, j)}_{=C_l^* \text{ if } M \geq l+1} x^l.$$

This gives the desired asymptotics for $\psi(x)$ in 0. For $x \to \infty$ we proceed analogously: Using again $\zeta(a,x) = O(x^{1-a})$ we find

$$\psi(x) = Q_\infty(x) + \sum_{m=0}^{M} \zeta'_\eta(m+2\nu+1, x+1) C_m$$
$$+ \underbrace{\sum_{n=1}^{\infty} (n+x)^{-2\nu-1} \eta\big(T'T^{n-1}\big)^{-1} \left(\psi\left(1-\frac{1}{n+x}\right) - \sum_{m=0}^{M} \frac{C_m}{(n+x)^m} \right)}_{=O(x^{-(2\nu+1+M)})}$$

and (12) implies

$$\sum_{m=0}^{M} \zeta'_\eta(m+2\nu+1, x+1) C_m \underset{x\to\infty}{\sim}$$
$$\underset{x\to\infty}{\sim} \sum_{l=-1}^{\infty} \underbrace{\sum_{m=0}^{M} \frac{1}{m+2\nu} \sum_{r=0}^{l+1-m} \binom{r-2\nu-l-1}{r} \sum_{j=0}^{N-1} \eta(T'T^j)^{-1} C_m \, C(l+1-r-m, j)}_{=C_l^{*\prime} \text{ if } M \geq l+1} x^{-l-2\nu-1}.$$

This gives the desired asymptotics for $\psi(x)$ in ∞. The last claim is obvious.

□

Remark 6.2 (i) If $\psi(x) = o(x^{-\min(1,2\mathrm{Re}\,(\nu)+1)})$ for $x \to 0$, then $Q_0 = 0$ by periodicity, i.e., ψ is an eigenfunction for the *transfer operator*

$$\mathcal{L}_0\psi(x) \stackrel{\text{def}}{=} x^{-2\nu-1}\sum_{n=0}^{\infty}(n+x^{-1})^{-2\nu-1}\eta\big(T(T')^n\big)^{-1}\psi\left(1+\tfrac{1}{n+x^{-1}}\right)$$

for the eigenvalue 1. Moreover we then have $C^*_{-1} = 0$.

(ii) If $\psi(x) = o(x^{-\min(0,2\mathrm{Re}\,(\nu))})$ for $x \to \infty$, then $Q_\infty = 0$, i.e., ψ is an eigenfunction for the *transfer operator*

$$\mathcal{L}_\infty\psi(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty}(n+x)^{-2\nu-1}\eta\big(T'T^n\big)^{-1}\psi\left(1+\tfrac{1}{n+x}\right)$$

for the eigenvalue 1. Moreover we then have $C^{*'}_{-1} = 0$.

(iii) Suppose that $\frac{1}{2} < \mathrm{Re}\nu$ and $\psi \in \Psi^{\mathbb{R}}_{\nu,\eta}$. Then (4) implies

$$(x+1)^{+2\nu+1}\big(\eta(T)\psi(x) - \psi(x+1) = \eta(ST^{-1})\psi\left(\frac{x}{x+1}\right),$$

and hence (5) implies that $C_0 = \psi(1) = 0$. By (i) and (ii) we have $Q_0 = Q_\infty = 0$, so we find the equations

$$\psi(x) = x^{-2\nu-1}\sum_{n=0}^{\infty}(n+x^{-1})^{-2\nu-1}\eta\big(T(T')^n\big)^{-1}\psi\left(1+\tfrac{1}{n+x^{-1}}\right) \quad (13)$$

$$\psi(x) = \sum_{n=1}^{\infty}(n+x)^{-2\nu-1}\eta\big(T'T^n\big)^{-1}\psi\left(1-\tfrac{1}{n+x}\right). \quad (14)$$

In this case, we can analytically extend ψ to $\mathbb{C}\setminus]-\infty,0]$ via

$$\psi(z) \stackrel{\text{def}}{=} \sum_{\gamma\in Q_n}(\psi|_{\nu,\eta}\gamma)(z),$$

where Q is the semigroup generated by T and T', Q_n is the set of T–T'-words of length n in Q, and

$$(\psi|_{\nu,\eta}\gamma)(z) \stackrel{\text{def}}{=} (cz+d)^{-2\nu-1}\eta(\gamma)^{-1}\psi(\gamma\cdot z)$$

is a well defined right semigroup action (cf. [12], § 3, and [16], §III.3). The analytically continued function ψ still satisfies (13) and (14). Therefore we can mimick the proof of Lemma 6.1 and use the Taylor expansion in 1 to find

$$\psi(z) = \sum_{m=0}^{M} z^{-2\nu-1}\zeta_\eta(m+2\nu+1, z^{-1})C_m + O\big(|\zeta(2\nu+M+2, z^{-1})|\big)$$

for $|z| \to 0$ and

$$\psi(z) = \sum_{m=0}^{M} \zeta'_\eta(m+2\nu+1, z+1)C_m + O\big(|\zeta(2\nu+M+2, z)|\big),$$

for $|z| \to \infty$. Now we use the following version of the asymptotic expansion of the Hurwitz zeta function, which can be found in [11], § 1.18:

$$\begin{aligned}\zeta(a,z) = {} & z^{1-a}\frac{\Gamma(a-1)}{\Gamma(a)} + \frac{1}{2}z^{-a} \\ & + \sum_{n=1}^{M} B_{2n}\frac{\Gamma(a+2n-1)}{\Gamma(a)(2n)!}z^{1-2n-a} + O(|z|^{-2M-1-a}) \end{aligned} \tag{15}$$

for $\mathrm{Re}(a) > 1$ and $z \in \mathbb{C}\setminus]-\infty, 0]$. Then we get

$$\psi(z) = O(1) \quad \text{for } |z| \to 0 \tag{16}$$

and, since the first terms of the asymptotic expansions of $\zeta(a,z)$ and $\zeta(a,z+1)$ agree,

$$\psi(z) = O(|z|^{-2\nu-1}) \quad \text{for } |z| \to \infty. \tag{17}$$

$\square$

Remark 6.3 One can use the slash action above to rewrite the real version (4) of the Lewis equation in the form

$$\psi = \psi|_{\nu,\eta}T + \psi|_{\nu,\eta}T'.$$

$\square$

Theorem 6.4 (cf. [16], Thm. 2) *Suppose that* $\mathrm{Re}(\nu) > -\frac{1}{2}$. *Then*

$$\Psi^{\mathbb{R}}_{\nu,\eta} = \{\psi|_{(0,\infty)} : \psi \in \Psi^{o}_{\nu,\eta}\}.$$

Proof: Note first that property (3) of $\psi \in \Psi^{o}_{\nu,\eta}$ trivially implies (5) and (6) for $\psi|_{(0,\infty)}$. Therefore it only remains to show that each element of $\Psi^{\mathbb{R}}_{\nu,\eta}$ occurs as the restriction of some $\psi \in \Psi^{o}_{\nu,\eta}$. To this end we fix a $\tilde{\psi} \in \Psi^{\mathbb{R}}_{\nu,\eta}$. Since (5) and (6) hold for $\tilde{\psi}$, we can apply Remark 6.2 to it. Thus $\tilde{\psi}$ has an analytic continuation to $\mathbb{C} \setminus (-\infty, 0]$ (still denoted by $\tilde{\psi}$) and the asymptotics (16) and (17) shows that $\tilde{\psi}$ indeed satisfies (3). $\square$

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